Examples of discrete probability distributions:

The binomial and Poisson distributions

Binomial Probability Distribution

- A fixed number of observations (trials), n
  - e.g., 15 tosses of a coin; 20 patients; 1000 people surveyed
- A binary random variable
  - e.g., head or tail in each toss of a coin; defective or not defective light bulb
  - Generally called “success” and “failure”
  - Probability of success is p, probability of failure is 1 – p
- Constant probability for each observation
  - e.g., Probability of getting a tail is the same each time we toss the coin

Binomial example

Take the example of 5 coin tosses. What's the probability that you flip exactly 3 heads in 5 coin tosses?

Binomial distribution

Solution:

One way to get exactly 3 heads: HHHTT

What's the probability of this <u>exact</u> arrangement?

P(heads)xP(heads) xP(heads)xP(tails)xP(tails) $=(1/2)^3 \, x \, (1/2)^2$

Another way to get exactly 3 heads: THHHT

Probability of this exact outcome = $(1/2)^1 \, x \, (1/2)^3 \, x \, (1/2)^1 = (1/2)^3 \, x \, (1/2)^2$

Binomial distribution

In fact, $(1/2)^3 \times (1/2)^2$ is the probability of each unique outcome that has exactly 3 heads and 2 tails.

So, the overall probability of 3 heads and 2 tails is:

$(1/2)^3 \times (1/2)^2 + (1/2)^3 \times (1/2)^2 + (1/2)^3 \times (1/2)^2$ + ….. for as many unique arrangements as there are—but how many are there??

Outcome	Probability
THHHT	$(1/2)^3$ *x* $(1/2)^2$
HHHTT	$(1/2)^3$ *x* $(1/2)^2$
TTHHH	$(1/2)^3$ *x* $(1/2)^2$
HTTHH	$(1/2)^3$ *x* $(1/2)^2$
HHTTH	$(1/2)^3$ *x* $(1/2)^2$
THTHH	$(1/2)^3$ *x* $(1/2)^2$
HTHTH	$(1/2)^3$ *x* $(1/2)^2$
HHTHT	$(1/2)^3$ *x* $(1/2)^2$
THHTH	$(1/2)^3$ *x* $(1/2)^2$
HTHHT	$(1/2)^3$ *x* $(1/2)^2$

10 arrangements *x* $(1/2)^3$ *x* $(1/2)^2$

$\binom{5}{3}$ ways to arrange 3 heads in 5 trials

$_5C_3 = 5!/3!2! = 10$

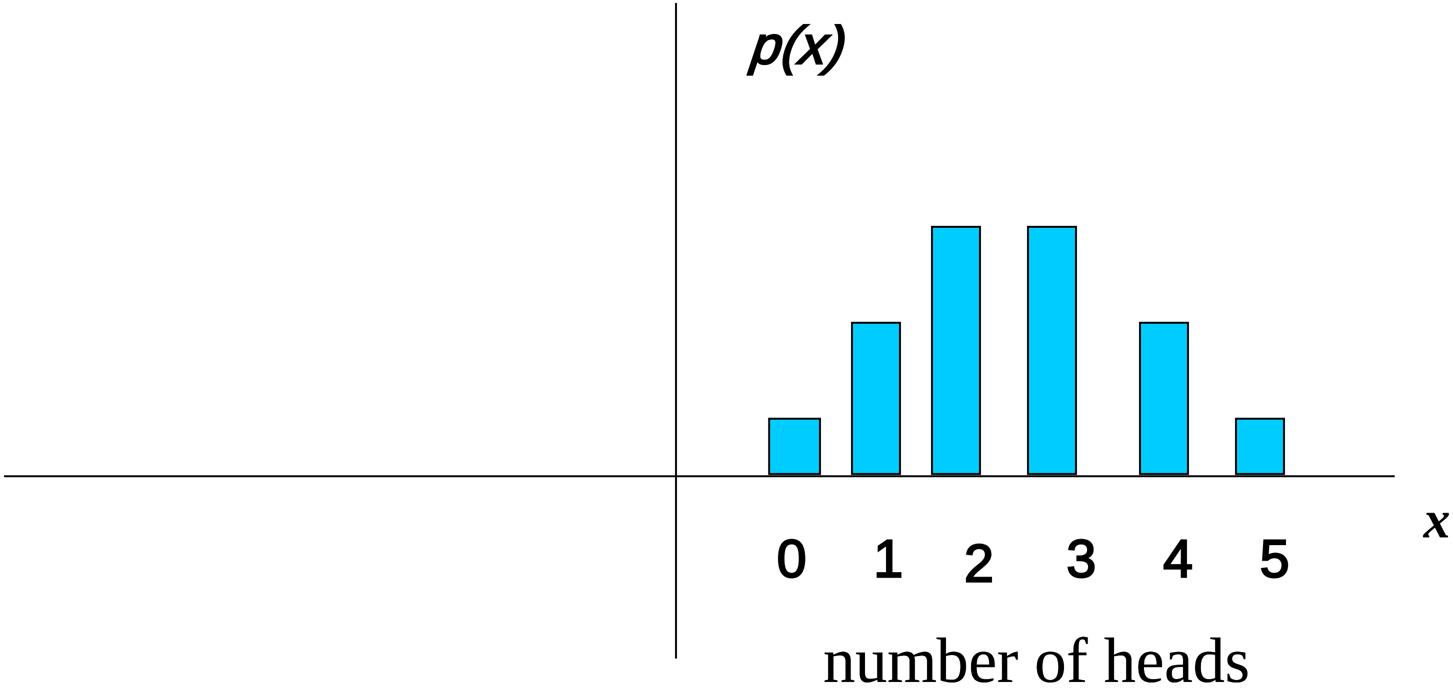
The probability of each unique outcome (note: they are all equal)

$\therefore$ **P(3 heads and 2 tails) =** $\binom{5}{3}$ ***x P(heads)^3 x P(tails)^2 =***

10 x (½)$^{5=}$31.25%

Binomial distribution function:

X = the number of heads tossed in 5 coin tosses



Example 2

As voters exit the polls, you ask a representative random sample of 6 voters if they voted for proposition 100. If the true percentage of voters who vote for the proposition is 55.1%, what is the probability that, *in your sample*, exactly 2 voted for the proposition and 4 did not?

Solution:

Outcome		Probability
YYNNNN		$= (.551)^2 \times (.449)^4$
NYYNNN	$(.449)^1 \times (.551)^2 \times (.449)^3$	$= (.551)^2 \times (.449)^4$
NNYYNN	$(.449)^2 \times (.551)^2 \times (.449)^2$	$= (.551)^2 \times (.449)^4$
NNNYYN	$(.449)^3 \times (.551)^2 \times (.449)^1$	$= (.551)^2 \times (.449)^4$
NNNNYY	$(.449)^4 \times (.551)^2$	$= (.551)^2 \times (.449)^4$
.		
.		

$\binom{6}{2}$ ways to arrange 2 Obama votes among 6 voters

15 arrangements x $(.551)^2 \times (.449)^4$

$$\therefore P(\text{2 yes votes exactly}) = \binom{6}{2} x\ (.551)^2 \times (.449)^4 = 18.5\%$$

Binomial distribution, generally

Note the general pattern emerging → if you have only two possible outcomes (call them 1/0 or yes/no or success/failure) in *n* independent trials, then the probability of exactly *X* "successes"=

n = number of trials

$$\binom{n}{X} p^X (1-p)^{n-X}$$

X = # successes out of *n* trials

p = probability of success

1-p = probability of failure

Definitions: Binomial

- **Binomial:** Suppose that n independent experiments, or trials, are performed, where n is a fixed number, and that each experiment results in a "success" with probability p and a "failure" with probability $1-p$. The total number of successes, X, is a binomial random variable with parameters n and p.
- We write: ***X ~ Bin (n, p)*** *{reads: "X is distributed binomially with parameters n and p}*
- And the probability that $X=r$ (i.e., that there are <u>exactly</u> *r* successes) is:

$$P(X=r)=\binom{n}{r}p^r(1-p)^{n-r}$$

Definitions: Bernouilli

Bernouilli trial: If there is only 1 trial with probability of success *p* and probability of failure *1-p*, this is called a Bernouilli distribution. (special case of the binomial with *n*=1)

Probability of success: $P(X=1)=\binom{1}{1}p^1(1-p)^{1-1}=p$

Probability of failure: $P(X=0)=\binom{1}{0}p^0(1-p)^{1-0}=1-p$

Binomial distribution: example

- If I toss a coin 20 times, what's the probability of getting exactly 10 heads?

$$\binom{20}{10}(.5)^{10}(.5)^{10} = .176$$

Binomial distribution: example

- If I toss a coin 20 times, what's the probability of getting of getting 2 or fewer heads?

$$\binom{20}{0}(.5)^0(.5)^{20} = \frac{20!}{20!0!}(.5)^{20} = 9.5x10^{-7} +$$

$$\binom{20}{1}(.5)^1(.5)^{19} = \frac{20!}{19!1!}(.5)^{20} = 20x9.5x10^{-7} = 1.9x10^{-5} +$$

$$\binom{20}{2}(.5)^2(.5)^{18} = \frac{20!}{18!2!}(.5)^{20} = 190x9.5x10^{-7} = 1.8x10^{-4}$$

$$= 1.8x10^{-4}$$

**All probability distributions are characterized by an expected value and a variance:

<u>If</u> *X* follows a binomial distribution with parameters *n* and *p*: ***X ~ Bin (n, p)***

<u>Then:</u>

$\mu_x = E(X) = np$

$\sigma_x^2 = Var(X) = np(1-p)$

$\sigma_x = SD(X) = \sqrt{np(1-p)}$

Note: the variance will always lie between

0*N-.25 *N

p(1-p) reaches maximum at p=.5

P(1-p)=.25

Introduction to the **Poisson Distribution**

- Poisson distribution is for counts—if events happen at a constant rate over time, the Poisson distribution gives the probability of X number of events occurring in time T.

Poisson Mean and Variance

- Mean

$$\mu = \lambda$$

- Variance and Standard Deviation

$$\sigma^2 = \lambda$$

$$\sigma = \sqrt{\lambda}$$

For a Poisson random variable, the variance and mean are the same!

where λ = expected number of hits in a given time period

Poisson Distribution, example

The Poisson distribution models counts, such as the number of new cases of SARS that occur in women in New England next month.

The distribution tells you the probability of all possible numbers of new cases, from 0 to infinity.

If X= # of new cases next month and X ~ Poisson (λ), then the probability that $X=k$ (a particular count) is:

$$p(X=k)=\frac{\lambda^k e^{-\lambda}}{k!}$$

Example

- For example, if new cases of West Nile Virus in New England are occurring at a rate of about 2 per month, then these are the probabilities that: 0,1, 2, 3, 4, 5, 6, to 1000 to 1 million to… cases will occur in New England in the next month:

Poisson Probability table

X	P(X)
0	$\frac{2^0 e^{-2}}{0!}$ =.135
1	$\frac{2^1 e^{-2}}{1!}$ =.27
2	$\frac{2^2 e^{-2}}{2!}$ =.27
3	$\frac{2^3 e^{-2}}{3!}$ =.18
4	=.09
5	
…	…

Example: Poisson distribution

Suppose that a rare disease has an incidence of 1 in 1000 person-years. Assuming that members of the population are affected independently, find the probability of k cases in a population of 10,000 (followed over 1 year) for k = 0,1,2.

The expected value (mean) =λ = .001*10,000 = 10
10 new cases expected in this population per year→

$$P(X=0)=\frac{(10)^0 e^{-(10)}}{0!}=.0000454$$

$$P(X=1)=\frac{(10)^1 e^{-(10)}}{1!}=.000454$$

$$P(X=2)=\frac{(10)^2 e^{-(10)}}{2!}=.00227$$

more on Poisson...

"Poisson Process" (rates)

Note that the Poisson parameter λ can be given as the mean number of events that occur in a defined time period OR, equivalently, λ can be given as a rate, such as λ=2/month (2 events per 1 month) that must be multiplied by t=time (called a "Poisson Process") →

X ~ Poisson (λ)

$$P(X=k)=\frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

E(X) = λt

Var(X) = λt

Example

For example, if new cases of West Nile in New England are occurring at a rate of about 2 per month, then what's the probability that exactly 4 cases will occur in the next 3 months?

X ~ Poisson (λ=2/month)

$$P(X = 4 \text{ in } 3 \text{ months}) = \frac{(2*3)^4 e^{-(2*3)}}{4!} = \frac{6^4 e^{-(6)}}{4!} = 13.4\%$$

Exactly 6 cases?

$$P(X = 6 \text{ in } 3 \text{ months}) = \frac{(2*3)^6 e^{-(2*3)}}{6!} = \frac{6^6 e^{-(6)}}{6!} = 16\%$$

Practice problems

1a. If calls to your cell phone are a Poisson process with a constant rate $\lambda=2$ calls per hour, what's the probability that, if you forget to turn your phone off in a 1.5 hour movie, your phone rings during that time?

1b. How many phone calls do you expect to get during the movie?

Answer

1a. If calls to your cell phone are a Poisson process with a constant rate λ=2 calls per hour, what's the probability that, if you forget to turn your phone off in a 1.5 hour movie, your phone rings during that time?

X ~ Poisson (λ=2 calls/hour)

$P(X\geq 1)=1 - P(X=0)$

$$P(X=0)=\frac{(2*1.5)^0 e^{-2(1.5)}}{0!}\frac{(3)^0 e^{-3}}{0!}=e^{-3}=.05$$

$\therefore P(X\geq 1)=1 - .05 = 95\%$ chance

1b. How many phone calls do you expect to get during the movie?

$E(X) = \lambda t = 2(1.5) = 3$